INFINITE FAMILIES OF SURFACE SYMMETRIES

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Dedicated to A. M. Macbeath with much respect

ABSTRACT

Consider a sequence $N_{a,b} : g \mapsto ag + b$, $g = 2, 3, \ldots$ where a, b are rational numbers and $a \geq 0$. We say that $N_{a,b}$ is **admissible** if for infinitely many g's there exists a finite group G of orientation-preserving homeomorphisms of a compact orientable surface of genus g such that $|G| = N_{a,b}(g)$. In this case we shall also say that G **belongs to** $N_{a,b}$. The main result of the paper is that if $a + b \neq 0$, then any group belonging to $N_{a,b}$ contains a cyclic subgroup of bounded index, where the bound depends only on a, b. Moreover $a + b$ is positive, and $2b/a$ is an integer. This implies, for instance, that there are only finitely many possibilities for perfect subgroups or quotient groups of groups belonging to $N_{a,b}$.

1. Introduction

(1.1) Let Σ_g denote a compact orientable surface of genus g. We shall always assume that g is at least 2. By a **symmetry group** of Σ_g we simply mean a finite group of orientation-preserving homeomorphisms of Σ_g. It is a well known consequence of Hurwitz's theory, cf. [Hu], that (for $g \geq 2$) there are only finitely many symmetry groups of Σ_g and their orders are bounded by $84(g-1)$. The symmetry groups of order $84(g-1)$ are precisely the finite quotients of the $\{2,3,7\}$ triangle group. More generally the finite quotients of a fixed cocompact Fuchsian group Γ have orders of the form $\frac{2}{-\chi(\Gamma)}(g-1)$ where $\chi(\Gamma)$ is the Euler

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characteristic of Γ in the sense of Wall, cf. [W]. There are infinitely many such quotients. This is the well known Miller–Nielsen–Fox result; for a simple proof see [K], §2. There are other examples of infinite families of surface symmetries of a different sort. For example, by a theorem of Wiman Σ_g always admits a cyclic symmetry group of order $4g+2$, cf. [Wi], [H]. Also it follows by a theorem of Accola and independently Maclachlan that Σ_g always admits a symmetry group of order $8g+8$, cf. [A], [M]. These families are not associated with any fixed Fuchsian group. A purpose of this note is to understand the nature of these other families.

(1.2) Let a, b be two rational numbers. Consider the sequence $N_{a,b} : g \mapsto ag+b, g = 2, 3 \ldots$. We say that $N_{a,b}$ is **admissible** if for infinitely many g's there exists a symmetry group G of order $N_{a,b}(g)$ acting on Σ_g. In this case we shall say that the pair (Σ_g, G), or more loosely G, or Σ_g **belongs to** $N_{a,b}$. Our main result is summarized in the following theorem.

THEOREM: (1) *Let $N_{a,b}$ be admissible and $a+b=0$. Then there exist finitely many (topological) cocompact Fuchsian groups Γ_i such that, except for finitely many exceptional values of g, the groups belonging to $N_{a,b}$ are precisely the ones uniformized by Γ_i's.*
(2) *Let $N_{a,b}$ be admissible and $a+b \neq 0$. Then any group belonging to $N_{a,b}$ contains a cyclic subgroup of bounded index, where the bound depends only on a, b. Moreover $a+b$ is positive, and $2b/a$ is an integer.*

Here by a **topological cocompact Fuchsian group** we mean that the complex structure or hyperbolic geometry needed in the usual definition of a Fuchsian group, cf. [B], is not relevant. We may as well define it to be a cocompact properly discontinuous group of orientation-preserving homeomorphisms of the plane with negative Euler characteristic.

Some simple consequences of this theorem to the structure of the groups in an infinite family $N_{a,b}$ with $a+b \neq 0$ are noted in Section 3. In a forthcoming joint work with M. Conder, cf. [CK], we shall make a larger input from finite group theory and construct several such infinite families and gather some quantitative information about these families. In particular we shall prove that in the above theorem, if $a+b \neq 0$ and $ag+b > 4(g-1)$ for all sufficiently large g, then b must be non-negative and a must be 4, 5, 6 or 8 or else it is a rational number of the form $4c/(c-1)$ for some integer c where $c = 4$ or $c \geq 6$. In the latter cases a is not an integer.

2. Proof of the Theorem

First suppose only that $N_{a,b}$ is admissible. We show that there are bounds depending only on a, b for the genus h of the quotient surface and the number r of the branch points for any group belonging to $N_{a,b}$. Indeed the Riemann–Hurwitz formula for a group of order $ag + b$ acting on Σ_g may be written in the form

$$\frac{2g-2}{ag+b} = 2h - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{n_i}\right). \tag{2.1.1}$$

Since n_i's are at least 2 the right-hand side is at least $2h-2+r/2$. As g tends to infinity the left-hand side tends to $2/a$. It is clear that h and r are bounded and so there are only finitely many possibilities for them. For the following discussion, passing to a subsequence if necessary we shall assume that h and r are actually fixed.

CASE 1: $a + b = 0$. We claim that the branching indices n_i's also have only finitely many possibilities. So in the first place we may assume that $r > 0$. Since $a + b = 0$, the equation (2.1.1) reduces to

$$\sum_{i=1}^{r} \frac{1}{n_i} = 2h - 2 + r - \frac{2}{a}. \tag{2.1.2}$$

Here the right-hand side is constant. Since the left-hand side is positive this constant is positive. Taking n_i's in an increasing order we see that the left-hand side is at most r/n_1. This shows that n_1 has only finitely many possibilities. Continuing this way we see that all n_i's have only finitely many possibilities. It follows that for sufficiently large g the groups belonging to $N_{a,b}$ are just the ones uniformized by the finitely many topological Fuchsian groups whose signatures are determined by the various but finitely many choices for h, r, and the n_i's.

Conversely, as was already noted earlier, groups uniformized by a Fuchsian group belong to an admissible sequence $N_{a,b}$ with $a + b = 0$.

CASE 2: $a+b\neq 0$. By the last remark it follows that for groups belonging to a sequence with $a+b\neq 0$, at least one of the branching indices must tend to ∞ as g tends to ∞. By passing to a subsequence if necessary we now assume that $h, r, n_1, \ldots, n_s$ are fixed and $n_{s+1}, \ldots, n_r$ tend to ∞ as g tends to ∞. So (2.1.1) may be written in the form

$$(2.1.3) \qquad \sum_{i=s+1}^{r} \frac{1}{n_i} + \left\{2h-2+r-\frac{2}{a}-\sum_{i=1}^{s}\frac{1}{n_i}\right\} + \frac{2(a+b)}{a(ag+b)}.$$

As g tends to ∞ the left-hand side and the last term on the right-hand side tend to 0. So the constant term $\{**\}$ in the above equation must be 0. Hence (2.1.3) implies

$$(2.1.4) \qquad a = 2\Big/\left\{2h-2+r-\sum_{i=1}^{s}\frac{1}{n_i}\right\}; \quad \sum_{i=s+1}^{r}\frac{1}{n_i} = \frac{2(a+b)}{a(ag+b)}.$$

Now n_i's are orders of elements in a group of order $ag+b$. So they are divisors of $ag+b$. Write $n_i u_i = ag+b$. Then (2.1.4) implies

$$(2.1.5) \qquad \sum_{i=s+1}^{r} u_i = \frac{2(a+b)}{a}.$$

In other words u_i's are bounded for $i > s$. It follows that the corresponding group has a cyclic subgroup of index bounded by a bound depending only on a, b. For large g we may take the bound to be $2(a+b)/a$. Also from (2.1.5) evidently $a+b>0$ and $2b/a$ is an integer. ∎

3. Some Consequences

(3.1) Throughout this section we fix two rationals a, b and suppose that $a+b\neq 0$ and the sequence $N_{a,b}$ is admissible. In this section we shall note some common properties of the symmetry groups belonging to $N_{a,b}$. Let G be a symmetry group belonging to $N_{a,b}$. Then there exists an integer d depending only on a, b such that G contains a cyclic subgroup, say A, of index at most d. Consider the permutation representation ρ of G on G/A. Let $\rho(G) = H$, and $A_0 = \ker\rho$. Then we have a short exact sequence

$$(3.1.1) \qquad 1 \longrightarrow A_0 \longrightarrow G \longrightarrow H \longrightarrow 1.$$

We shall use this notation in the following. Also we set $G^0 = G$, $G^1 = [G, G]_1$ $\ldots$, $G^k = [G^{k-1}, G^{k-1}]$, etc.

(3.2) PROPOSITION: *There exist constants ℓ and m depending only on a, b such that for all symmetry groups G belonging to $N_{a,b}$ we have $|G^\ell| \le m$.*

Proof: In the notation of (3.1) choose k such that $H^k = H^{k+1}$, i.e. H^k is perfect, possibly trivial. Then we have a short exact sequence

$$1 \longrightarrow A_0 \cap G^k \longrightarrow G^k \longrightarrow H^k \longrightarrow 1. \tag{3.2.1}$$

Now $A_0 \cap G^k$ is cyclic and so its automorphism group is abelian. Also the conjugation action of G^k on $A_0 \cap G^k$ factors through H^k. Since H^k is perfect it follows that its action on $A_0 \cap G^k$ is trivial, i.e. $A_0 \cap G^k$ lies in the center of G^k. Let U be the universal central extension of H^k given by the Schur theory, cf. [Mi], §5. So there is a homomorphism $\phi : U \to G^k$ such that $\rho \circ \phi$ is surjective onto H^k. Let U_0 be the image of U in G^k. Then U_0 is perfect (possibly trivial) and clearly G^k is generated by $A_0 \cap G^k$ and U_0. Since $A_0 \cap G^k$ lies in the center of G^k it easily follows that $G^{k+1} = [U_0, U_0] = U_0$. Now the order of H is at most $d!$ where d is the index of A in G. So there are only finitely many possibilities for H, hence only finitely many possibilities for the perfect groups H^k and their universal central extensions U all depending only on a, b. So the order of G^{k+1} is bounded by a bound depending only on a, b. ∎

(3.3) COROLLARY: *The length of the derived series of any solvable subgroup of a symmetry group belonging to $N_{a,b}$ is bounded by a bound depending only on a, b.*

(3.4) COROLLARY: *The order of a perfect subgroup or a perfect quotient group of a symmetry group belonging to $N_{a,b}$ is bounded by a bound depending only on a, b.*

(3.5) COROLLARY: *The product of the orders of nonabelian factors of a maximal composition series of a symmetry group belonging to $N_{a,b}$ is bounded by a bound depending only on a, b.*

(3.6) Recall that a p-rank of a finite group is the maximum rank of its elementary abelian p-subgroup.

PROPOSITION: *The p-ranks of a symmetry group belonging to $N_{a,b}$ are bounded by a bound depending only on a, b and independent of p.*

Proof: Indeed in the notation of (3.1) the p-rank of G is at most 1+ the p-rank of H, and there are only finitely many possibilities for H depending only on a, b. ∎

(3.7) For some interesting families it happens that only one of the branching indices corresponding to the symmetry groups belonging to $N_{a,b}$ tends to infinity and the quotient surface has genus 0, i.e. in the notation of (2.1.3) we have $s = r - 1$, and $h = 0$.

PROPOSITION: *If only one of the branching indices corresponding to the symmetry groups belonging to $N_{a,b}$ tends to infinity and the quotient surface has genus 0, then except for finitely many exceptions the symmetry groups belonging to $N_{a,b}$ cannot be cyclic-by-perfect.*

Proof: Suppose this is not the case. We know by (3.4) that there are only finitely many possibilities for perfect quotients for symmetry groups belonging to $N_{a,b}$. Consider the infinite subfamily of G's where the quotient is a fixed perfect group P. By the argument in (3.2) these G's are central extensions of P and so $[G, G]$ is bounded by a bound depending only on a, b. On the other hand if only one of the branching indices corresponding to the symmetry groups belonging to $N_{a,b}$ tends to infinity and the quotient surface has genus 0, then all these groups are generated by a fixed finite number of elements of fixed finite orders. But then $G/[G, G]$ is an abelian group also generated by a fixed finite number of elements of fixed finite orders. So its order is bounded by a bound depending only on a, b. Since the order of both $[G, G]$ and $G/[G, G]$ is bounded by a bound depending only on a, b it follows that there are only finitely many possibilities for G's. This contradiction proves the assertion. ∎

(3.8) We remark that the hypothesis in (3.7) that only one of the branching indices corresponding to the symmetry groups belonging to $N_{a,b}$ tends to infinity cannot be omitted. Indeed we construct here an admissible sporadic $(4g + 2k)$-family for a suitable k. Let G be any finite group generated by two elements x, y such that the order of x is 2. Then for any natural number c which is coprime to the orders of y and xy consider the groups $G_c = G \times Z_c$, where Z_c is a cyclic group of order c. Let u be a generator of Z_c, and let ℓ resp. m be the order of y resp. xy. Then G_c is generated by two elements $(x, e), (y, u)$ which have orders 2 and $c\ell$ respectively. The product of these elements has order cm. So by Hurwitz's theory G_c is a symmetry group of Σ_g with the quotient of genus 0 and 3 branch points with branching indices $\{2, c\ell, cm\}$ respectively. Here g is calculated by the Riemann–Hurwitz formula:

$$2g - 2 = c|G| \left\{ \left(1 - \frac{1}{2}\right) - \frac{1}{c\ell} - \frac{1}{cm} \right\},$$

or

$$c|G| = 4g + \{2|G|/\ell + 2|G|/m - 4\}.$$

In other words setting

$$k = |G|/\ell + |G|/m - 2$$

we see that the groups G_c's belong to a $(4g + 2k)$-family.

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